NORTH-HOLLAND

Unfolding the Zero Structure of a Linear Control System

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ABSTRACT

This paper is motivated by the problem of controller design for a parameter-dependent linear system. Many compensation techniques depend critically on aspects of the zero structure of the system, such as relative degree, or the nonminimum-phase property. However, the zero structure is structurally unstable, meaning it may change discontinuously with small changes in the parameters. Thus it is crucial that the designer know what structures and structural transitions are possible. This paper uses a miniversal deformation, or unfolding, of the Kronecker form previously reported by the authors, and applies it directly to pencils in a canonical form of the system matrix. The unfolding is used to explore all zero structures in the neighborhood of a nominal system. Several examples are presented. When possible, the results are presented in the form of a bifurcation diagram. © Elsevier Science Inc., 1997

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1. INTRODUCTION

This paper considers linear, time-invariant, parameter-dependent systems

$$\dot{x} = A(\mu)x + B(\mu)u, \tag{1.1a}$$

$$y = C(\mu)x + D(\mu)u. \tag{1.1b}$$

Matrix pencils are matrix-valued functions of a scalar variable of the form $M(s) = M_1 + M_2 s$. Two pencils $M(s)$ and $N(s)$ are said to be strictly equivalent (s.e.) if there exist constant, invertible matrices G_1 and G_2 such that $N_1 = G_1 M_1 G_2^{-1}$ and $N_2 = G_1 M_2 G_2^{-1}$. The system (1.1) is associated with a parameter-dependent matrix pencil, commonly called the *system matrix*,

$$\Gamma_\mu(s) = \begin{bmatrix} sI - A(\mu) & B(\mu) \\ -C(\mu) & D(\mu) \end{bmatrix}. \tag{1.2}$$

The use of the system matrix and related pencils in control theory has been extensively studied by Van Dooren (1981). Many equivalence relations between system matrices are of interest in control theory. Among them are those preserving the system transfer function (Rosenbrock, 1974; Fuhrmann, 1977) and those preserving only the zero structure while destroying the pole structure (Kouvaritakis and MacFarlane, 1976; Owens, 1978). The various definitions of zeros are linked to different equivalence relations. Two surveys of interest on this topic are by MacFarlane and Karcanias (1976) and by Schrader and Sain (1989). For the remainder of this paper, *zero structure* will refer to the invariants of (1.2) under s.e. transformation. The Kronecker form is the classical choice of canonical forms under this equivalence relation (Gantmacher, 1959). The invariants are given various control-system interpretations by Morse (1973), Thorp (1973), and Molinari (1978).

The above definition of zero structure in turn leads to a definition of structural instability: A system is *structurally unstable* if an arbitrarily small perturbation changes its invariants under strict equivalence. Exactly which perturbations are to be allowed will be discussed extensively later. A system that is not structurally unstable is called *structurally stable*.

Consider the problem of controlling a family of systems, defined by an analytically parameter-dependent linear state equation. In such situations, structurally unstable systems may arise unavoidably, in the sense that all neighboring families also contain such a member. These isolated members of a family are extremely interesting, even though they are unlikely to be observed in practice, because they organize the parameter space into distinct partitions. If the choice of structure has been made properly, then these partitions should have significance to the compensation scheme. An example of regulator design which is sensitive to zero location is given in this paper.

The relative degree is also an invariant of s.e. transformation, and can be extremely sensitive to perturbation. This has interesting consequences in systems with relative degree greater than one, which are to be controlled using dynamic inversion. Small perturbations may cause the zeros at infinity to become finite (though large), and possibly nonminimum phase. Nonminimum-phase systems cannot be controlled using standard dynamic inversion techniques, though there is no problem with high relative degree. This was first exploited in the well-known example given by Lane and Stengel for aircraft (1988). In that work, a system is artificially forced to be structurally unstable (in this instance, to have relative degree greater than one) for design purposes, rather than exhibit the true, but unacceptable, property (nonminimum-phase zeros).

The standard tool for charting the structures to be found in the neighborhood of a structurally unstable system is the miniversal deformation (Arnold, 1981, 1983). This is a parameter-dependent normal form that uses the fewest possible parameters to reach all nearby structures. A miniversal deformation of a singular point is also referred to here as an unfolding, following the nomenclature of bifurcation and singularity theory (Golubitsky and Guillemin, 1973; Guckenheimer and Holmes, 1983; Golubitsky and Schaeffer, 1984; Bruce and Giblin, 1984). An unfolding of Kronecker form has been presented by the authors (Berg and Kwatny, 1995). However, because the Kronecker form does not preserve the system matrix structure of a pencil, it is often not the best choice for the analysis of control systems. A more suitable canonical form, strictly equivalent to the Kronecker form, was suggested independently by Thorp (1973) and Morse (1973). The unfolding of the Kronecker form, appropriately transformed, is also an unfolding of the Thorp-Morse form. Several examples of the result are presented here. The actual transformation is straightforward, if somewhat labor-intensive, and a full listing of the result is omitted in the interest of brevity.

Miniversal deformations have been applied to several problems of control theory. Tannenbaum derives a miniversal deformation of systems under similarity transformation (1981) to examine the properties of uncontrollable and unobservable systems. The resulting miniversal deformations are unrelated to those presented here. More recently, researchers studying numerical analysis of matrix pencils have turned to versal deformations for insight into robust calculation of Kronecker invariants. Demmel and Edelman (1992) calculate the dimension of the miniversal deformation under s.e., but do not derive the deformation itself. Edelman, Elmroth, and Kagstrom (1995) do compute a miniversal deformation of the Kronecker form that has desirable features for numerical analysis. Their formulation has drawbacks for application to control systems, for reasons which will be discussed below. Ferrer and Garcia-Planas (1996) use equations defining a miniversal deformation under s.e. to determine the structural stability of the system (1.1), but they do not present the full unfolding.

An important issue not discussed here is how to find the structurally unstable elements of a family. In practice such points may be known, or they may be found numerically, using techniques such as those suggested by Elmroth and Kagstrom (1996).

Section 2 of this paper presents some necessary background. Section 3 reviews canonical forms of matrix pencils. Section 4 briefly discusses some aspects of the unfolding. Section 5 presents examples and applications. When possible the results are presented as bifurcation diagrams.

2. BACKGROUND AND NOTATION

2.1. *Notation*

Scalar variables are indicated with italic or Greek letters: k, K, or κ. Vectors and matrices are sans serif italic characters, lower- and uppercase respectively: x and A. O denotes both the zero vector and the zero matrix. $\mathsf{O}^{(m)(n)}$ denotes an $m \times n$ matrix of zeros. $\mathbf{0}$ is used in matrices as a "space-filling" zero. $\mathsf{I}^{(n)}$ denotes an $n \times n$ identity matrix. $\mathsf{H}^{(n)}$ denotes an $n \times n$ matrix with ones on the main superdiagonal, and $\mathsf{H}_{(n)}$ denotes an $n \times n$ matrix with ones on the main subdiagonal.

2.2. *Singularity and Unfoldings*

One may reasonably expect never to run across a particular structurally unstable system in practice. What, then, from the point of view of compensator design, makes such systems worthy of extensive study? The answer is, the versal deformation of a structurally unstable point shows precisely what structures may be encountered, what structural transitions are possible, and which singular structures will be persistent in parametrized families.

The *orbit* of a system is its equivalence class under s.e. These orbits are manifolds (Berg and Kwatny, 1995), and it is interesting to consider the codimension of those manifolds.

When the codimension is zero, the system is structurally stable; otherwise it is structurally unstable (Ferrer and Garcia-Planas, 1996). Now consider structurally unstable systems of codimension one. Figure 1 is a schematic in $\mathbb{R}^3$. M is a system, of codimension one, contained in the smooth, one

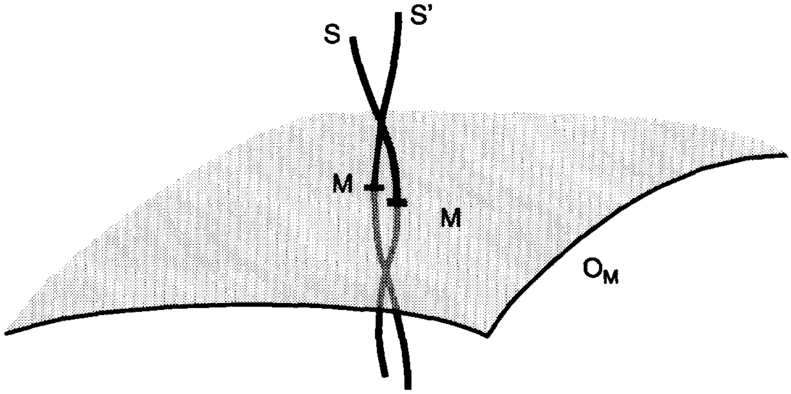

FIG. 1. One-parameter family containing codimension-one singular point.

parameter family S. the orbit of M, O_M, is a two-dimensional manifold of equivalent codimension one systems. M partitions S into three equivalence classes: two disjoint open sets of structurally stable systems, and a structurally unstable system. Because O_M and S intersect transversely, a slightly perturbed family, S', will also intersect O_M, now at a slightly different point M'. Since S' and S have the same partition structure, the transition between the two open sets will persist under small perturbation.

Consider now systems of higher codimension. Figure 2 shows a one-dimensional manifold O_M of equivalent codimension-two structurally unstable

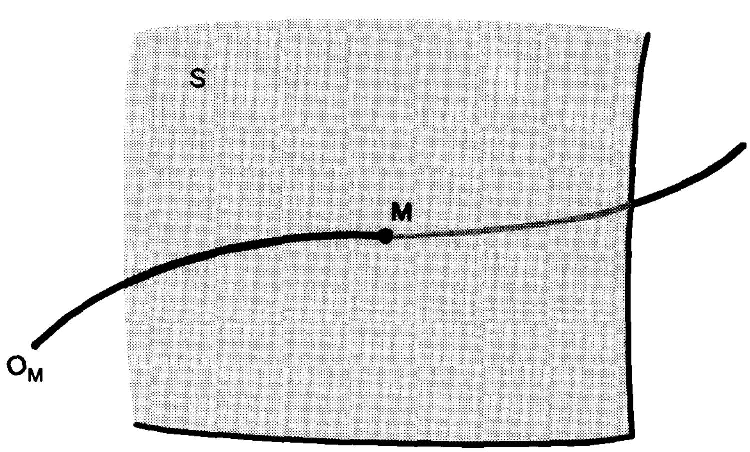

FIG. 2. Two-parameter family containing codimension-two singular point.

systems. Clearly it is not possible for any one-parameter family of systems to intersect this manifold transversely. A general two-dimension manifold S representing the family intersects the one-dimensional manifold O_M transversally, as shown schematicly in Figure 2. But the structurally unstable system contained in S does not partition the family, as it did for the codimension-one case. What then is its significance? The importance arises when the orbit of the higher-codimension system forms the boundary of *several* orbits of structurally unstable systems of codimension one. Figure 3 shows one possibility. Examining a neighborhood of S near the intersection point M with O_M reveals a partition into nine sets. Four of these are open quarter planes of structurally stable systems, four are curves consisting of codimension-one structurally unstable systems, and one is a single structurally unstable system of codimension two. Note that other arrangements are certainly possible. The higher-codimension systems may simply lie on the boundary between two lower-codimension orbits, perhaps on the edge of a cusp (Arnold, 1981).

A sketch such as Figure 3, along with the corresponding structures, is commonly called a bifurcation diagram. That nomenclature is adopted here. A possible objection to this usage is that the partitions separated by these

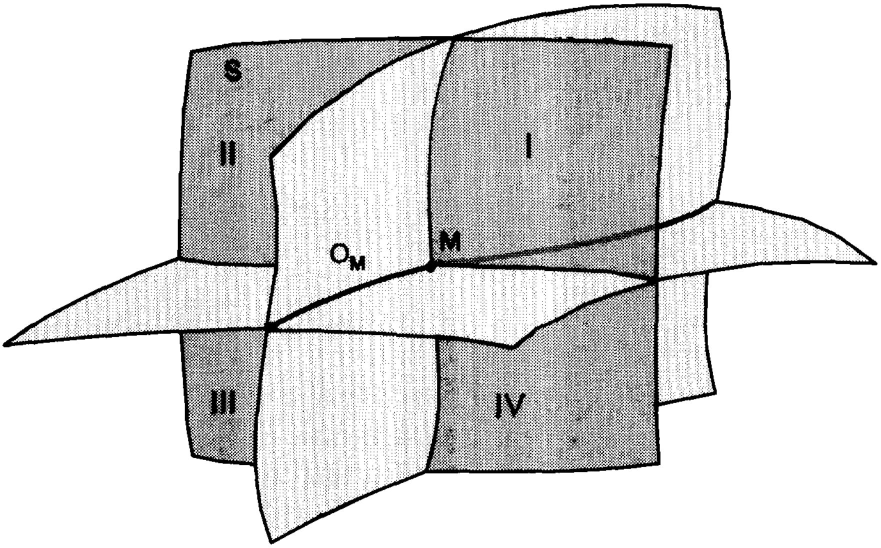

FIG. 3. Two codimension-one manifolds of singular points intersect to form a codimension two manifold of singular points. The result is a partition of S into nine pieces: the quarter-planes I, II, III, IV, the four lines forming the boundaries I-II, II-III, III-IV, and IV-I, and the point M itself.

boundaries do not necessarily have different zero structures. However, the results of studies by the authors (Kwatny et al., 1990; Berg and Kwatny, 1994), as well as the example in Section 5.1 of this paper, suggest that these partitions correspond to distinct *closed-loop* behaviors, and that the boundaries do represent true bifurcations of the *compensated* system.

2.3. *Miniversal Deformations*

Let $\mathcal{M}$ be an analytic manifold, and $\mathcal{G}$ be a Lie group, with identity element $\boldsymbol{I}$, that acts on $\mathcal{M}$ through $\boldsymbol{G} \cdot \boldsymbol{M} \mapsto \boldsymbol{N}$, where $\boldsymbol{M}, \boldsymbol{N} \in \mathcal{M}$, $\boldsymbol{G} \in \mathcal{G}$. Recall that a smooth map $\alpha : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, also written $\alpha_{\boldsymbol{M}}(\boldsymbol{G})$ or $\boldsymbol{G} \cdot \boldsymbol{M}$, is an *action* of $\mathcal{G}$ on $\mathcal{M}$ if (1) $\boldsymbol{I} \cdot \boldsymbol{M} = \boldsymbol{M}$ and (2) $\boldsymbol{G}_2 \cdot (\boldsymbol{G}_1 \cdot \boldsymbol{M}) = (\boldsymbol{G}_2 \cdot \boldsymbol{G}_1) \cdot \boldsymbol{M}$. The action defines an equivalence relation by $\boldsymbol{M} \sim \boldsymbol{N}$ if there exists some $\boldsymbol{G}$ such that $\boldsymbol{G} \cdot \boldsymbol{M} = \boldsymbol{N}$. Consider an element $\boldsymbol{A} \in \mathcal{A}$, where $\mathcal{A}$ can be either $\mathcal{M}$ or $\mathcal{G}$, and a parameter vector $\boldsymbol{c} \in \mathcal{C} \subset \mathbb{C}^k$. A mapping $\boldsymbol{A} : \mathcal{C} \to \mathcal{A}$, written $\boldsymbol{A}(\boldsymbol{c})$, is called a *deformation* of $\boldsymbol{A}$ if (1) the entries of $\boldsymbol{A}(\boldsymbol{c})$ are power series in the elements of c, convergent in some neighborhood of $\boldsymbol{c} = \boldsymbol{0}$, and (2) $\boldsymbol{A}(0) = \boldsymbol{A}$. The space $\mathcal{C}$ is called the *base* of the deformation. Now consider $\mathcal{M}$ and $\mathcal{G}$, and the action of $\mathcal{G}$ on $\mathcal{M}$. Two deformations of the same element of $\mathcal{M}$, $\boldsymbol{M}(\boldsymbol{c})$ and $\boldsymbol{N}(\boldsymbol{c})$ say, with the same base, are *equivalent* if there exists a deformation of the identity element of $\mathcal{G}$, $\boldsymbol{G}(\boldsymbol{c})$, with that base, such that $\boldsymbol{M}(\boldsymbol{c}) = \boldsymbol{G}(\boldsymbol{c}) \cdot \boldsymbol{N}(\boldsymbol{c})$. Note that $\boldsymbol{M}(\boldsymbol{0}) = \boldsymbol{N}(\boldsymbol{0}) = \boldsymbol{M}$ and $\boldsymbol{G}(\boldsymbol{0}) = \boldsymbol{I}$.

Next consider a second parameter vector $\boldsymbol{d} \in \mathcal{D} \subseteq \mathbb{C}^l$, and a mapping $\phi : \mathcal{D} \to \mathcal{C}$. Require that (1) the elements of ϕ be power series in the elements of $\boldsymbol{c}$, convergent in some neighborhood of $\boldsymbol{c} = \boldsymbol{0}$, and (2) $\phi(\boldsymbol{0}) = \boldsymbol{0}$. Then define the composition $\boldsymbol{M}(\phi(\boldsymbol{d}))$, the mapping *induced* by $\boldsymbol{M}(\boldsymbol{c})$ under ϕ.

DEFINITION. A deformation $\boldsymbol{M}(\boldsymbol{c})$ is called *versal* if any arbitrary deformation of the same pencil is equivalent to a deformation induced by $\boldsymbol{M}(\boldsymbol{c})$. That is, any $\boldsymbol{N}(\boldsymbol{d})$ can be written

$$\boldsymbol{N}(\boldsymbol{d}) = \boldsymbol{G}(\boldsymbol{d}) \cdot \boldsymbol{M}(\phi(\boldsymbol{d})) \tag{2.1}$$

with $\boldsymbol{M}(\boldsymbol{0}) = \boldsymbol{N}(\boldsymbol{0}) = \boldsymbol{M}$, $\boldsymbol{G}(\boldsymbol{0}) = \boldsymbol{I}$ [that is, $\boldsymbol{G}(\boldsymbol{d})$ is a deformation of the identity] and $\phi(\boldsymbol{0}) = \boldsymbol{0}$.

If in addition the dimension of $\mathcal{C}$ is minimal, the mapping is called *miniversal* (Arnold, 1981). The following theorem is invaluable for calculation, and provides useful geometric insight into versal deformations.

THE TRANSVERSALITY THEOREM (Tannenbaum, 1981). *A deformation* $\boldsymbol{M}(\boldsymbol{c})$ *is versal if and only if its image intersects the orbit of* $\boldsymbol{M}$ *transversally at* $\boldsymbol{M}$.

It is tempting to say that the miniversal deformation gives access to all nearby structures, with the smallest number of parameters, but, as the following example shows, that is not quite accurate. Consider the Lie group SO(2) of rotations in two dimensions, acting on the manifold $\mathbb{R}^2$. The orbit of a nonzero element of $\mathbb{R}^2$ is the circle, centered on the origin, passing through that point. The origin is equivalent only to itself, so its orbit is a single point. The situation is as shown in Figure 4.

Consider now a neighborhood of any nonzero element $\boldsymbol{P}_1$. The neighborhood is small enough that the orbits can be treated as straight lines. Note that the orbit of $\boldsymbol{P}_1$ has codimension one, and so the miniversal deformation of x should require one parameter. In fact, such a deformation is given by $\boldsymbol{P}_1(1 + \lambda)$, with tangent space spanned by $\boldsymbol{v} = \boldsymbol{P}_1$. Geometrically, any nearby point $\boldsymbol{P}_2$ can be reached from $\boldsymbol{P}_1$ by first moving radially to the orbit of $\boldsymbol{P}_2$.

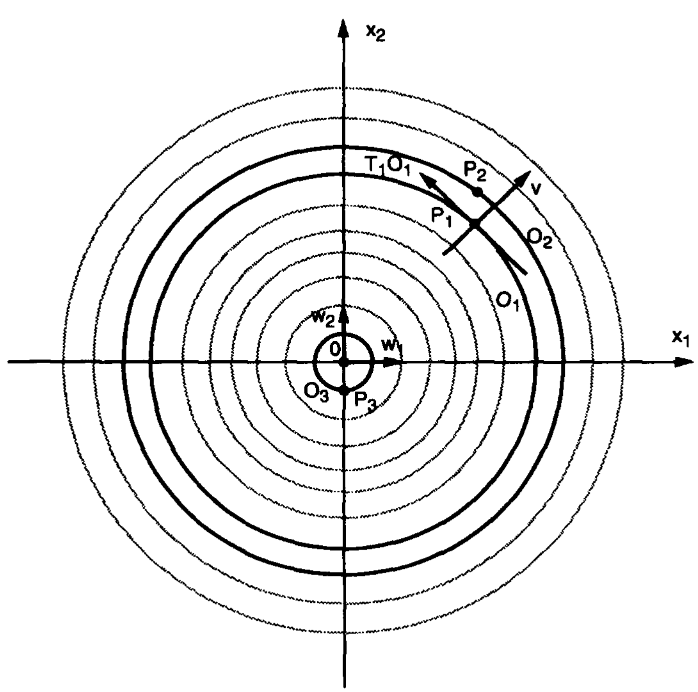

FIG. 4. The geometry of a simple example of miniversal deformations.

Then P_2 is reached by moving along the orbit. Structurally, any point on the orbit of P_2 is equivalent to P_2, and so a single radial degree of freedom is both necessary and sufficient to access all nearby structures.

Now consider a miniversal deformation of the origin. The orbit has codimension two, so two parameters are required. Any linear independent choice will do, say (w_1, w_2). Clearly this deformation is sufficient to reach any nearby point. However, it is easy to see that it is not necessary. As in the case of nonzero vectors, all that is required in a single radial degree of freedom. Unlike the first case, however, a large rotation may be required once the orbit is reached. As an example, P_3 in Figure 4 could be reached from the origin by moving along w_1 to the orbit O_3, then one quarter turn around the orbit to P_3. However, the definition of versality restricts the equivalence transformations to be in a small neighborhood of identity, which this large rotation is not. Therefore miniversal deformations may sometimes over-parametrize a point on the manifold.

The following result is required for this paper.

PROPOSITION. *Let $\boldsymbol{M}(\lambda)$ be a miniversal deformation of $\boldsymbol{M}$. Let $\boldsymbol{N}$ be equivalent to $\boldsymbol{M}$, with $\boldsymbol{N} = \boldsymbol{G} \cdot \boldsymbol{M}$. Then $\boldsymbol{N}(\lambda) \overset{\text{def}}{=} \boldsymbol{G} \cdot \boldsymbol{M}(\lambda)$ is a miniversal deformation of $\boldsymbol{N}$.*

Proof. First show that the versality of $\boldsymbol{M}(\lambda)$ implies the versality of $\boldsymbol{N}(\lambda)$. Let $\tilde{\boldsymbol{N}}(\xi)$ be an arbitrary deformation of $\boldsymbol{N}$. Consider $\boldsymbol{G}^{-1} \cdot \tilde{\boldsymbol{N}}(\xi)$. The entries of $\boldsymbol{G}^{-1} \cdot \tilde{\boldsymbol{N}}(\xi)$ are power series in ξ, and $\boldsymbol{G}^{-1} \cdot \tilde{\boldsymbol{N}}(0) = \boldsymbol{G}^{-1} \cdot \boldsymbol{N} = \boldsymbol{G}^{-1} \cdot \boldsymbol{G} \cdot \boldsymbol{M} = \boldsymbol{M}$. So $\boldsymbol{G}^{-1} \cdot \tilde{\boldsymbol{N}}(\xi)$ is a deformation of $\boldsymbol{M}$. Then, by the versality of $\boldsymbol{M}(\lambda)$, there exist $\boldsymbol{H}(\xi)$ and $\phi(\xi)$ such that $\boldsymbol{G}^{-1} \cdot \tilde{\boldsymbol{N}}(\xi) = \boldsymbol{H}(\xi) \cdot \boldsymbol{M}(\phi(\xi))$ with, in particular, $\boldsymbol{H}(0) = \boldsymbol{I}$. Write $\boldsymbol{M}(\lambda) = \boldsymbol{G}^{-1} \cdot \boldsymbol{N}(\lambda)$, and let $\boldsymbol{G}$ act on both sides of the equation to get $\tilde{\boldsymbol{N}}(\xi) = \boldsymbol{G} \cdot \boldsymbol{H}(\xi) \cdot \boldsymbol{G}^{-1} \cdot \boldsymbol{N}(\phi(\xi))$. Clearly $\boldsymbol{G} \cdot \boldsymbol{H}(\xi) \cdot \boldsymbol{G}^{-1}$ is a deformation of the identity, and so $\boldsymbol{N}(\lambda)$ is versal. Minimality follows easily by noting that if $\tilde{\boldsymbol{N}}(\xi)$ is a versal deformation of $\boldsymbol{N}$, then $\tilde{\boldsymbol{M}}(\xi) = \boldsymbol{G}^{-1} \cdot \tilde{\boldsymbol{N}}(\xi)$ is a versal deformation of $\boldsymbol{M}$. But if the base of $\tilde{\boldsymbol{N}}(\xi)$ has lower dimension than the base of $\boldsymbol{N}(\lambda)$, then the base of $\tilde{\boldsymbol{M}}(\xi)$ has lower dimension than the base of $\boldsymbol{M}(\lambda)$. This is impossible, by the minimality of $\boldsymbol{M}(\lambda)$, and so the deformation $\boldsymbol{N}(\lambda)$ is miniversal. ■

Berg and Kwatny have calculation a miniversal parametrization of the Kronecker form (Berg, 1992; Berg and Kwatny, 1995). Since the Kronecker form and the Thorp-Morse form are both canonical under s.e. transformation, this parametrization is easily adapted to the Thorp-Morse form via the preceding result. All that is required is to take the s.e. transformation from Kronecker form to Thorp-Morse form, and apply it to the miniversal deformation.

3. CANONICAL FORMS FOR THE SYSTEM MATRIX

The invariants associated with strict equivalence are as follows: Kronecker column (or right) indices, $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_h = 0 < \varepsilon_{h+1} \leqslant \varepsilon_{h+2} \leqslant \cdots \leqslant \varepsilon_p$; Kronecker row (or left) indices, $\eta_1 = \eta_2 = \cdots = \eta_g = 0 < \eta_{g+1} \leqslant \eta_{g+2} \leqslant \cdots \leqslant \eta_q$; degree of infinite divisors, $\rho_1 = \rho_2 = \cdots = \rho_r < \rho_{r+1} \leqslant \rho_{r+2} \leqslant \cdots \leqslant \rho_v$; a square matrix J in Jordan normal form, containing the finite zero structure. The classical canonical form for matrix pencils is the Kronecker form (Gantmacher, 1959). The Kronecker form clearly displays the invariants on the pencil, but it has the disadvantage, for use in systems theory, that the explicit system-matrix interpretation of the pencil is destroyed. An equivalent form that preserves the system-matrix structure, as well as allowing a feedback interpretation of the operations of strict equivalence transformation, has been presented by Thorp (1973) and (slightly less generally) by Morse (1973). The Thorp-Morse form is as follows:

$$\tilde{A} = \begin{bmatrix} \overbrace{A_{\infty\infty}}^{\sum_{i=r+1}^{v} \rho_i} & \overbrace{0}^{n} & \overbrace{0}^{\sum_{i=h+1}^{p} \epsilon_i} & \overbrace{0}^{\sum_{i=g+1}^{q} \eta_i} \\ 0 & A_{ff} & 0 & 0 \\ 0 & 0 & A_{\varepsilon\varepsilon} & 0 \\ 0 & 0 & 0 & A_{\eta\eta} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} \overbrace{B_{\infty\infty}}^{v-r} & \overbrace{0}^{h-p} & \overbrace{0}^{r} & \overbrace{0}^{g} \\ 0 & 0 & 0 & 0 \\ 0 & B_{\varepsilon\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \}\ \sum_{i=r+1}^{v} \rho_i \\ \}\ n \\ \}\ \sum_{i=h+1}^{p} \varepsilon_i \\ \}\ \sum_{i=g+1}^{q} \eta_i \end{matrix}$$

$$\tilde{C} = \begin{bmatrix} C_{\infty\infty} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\eta\eta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(r)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \} \; v - r \\ \} \; q + h \\ \} \; r \\ \} \; h \end{matrix}$$

Here $A_{ff} = -J$, and other blocks have the form $A_{\infty\infty} = \mathrm{diag}\{H_{(\rho_{r+i})}\}$ with $i = 1, \ldots, \alpha_\infty$; $A_{\varepsilon\varepsilon} = \mathrm{diag}\{H_{(\varepsilon_{h+i})}\}$ with $i = 1, \ldots, \alpha_\varepsilon$; and $A_{\eta\eta} = \mathrm{diag}\{H_{(\eta_{g+i})}\}$ with $i = 1, \ldots, \alpha_\eta$. The α_ι take the values $\alpha_\infty = v - r$, $\alpha_\varepsilon = p - h$, $\alpha_\eta = q - g$.

The corresponding nonzero partitions of $\tilde{B}$ and $\tilde{C}$ have the block-diagonal structure

$$B_{\iota\iota} = \begin{bmatrix} e_1 & & & \mathbf{0} \\ & e_2 & \ddots & \\ \mathbf{0} & & & e_{\alpha_l} \end{bmatrix}, \qquad C_{\iota\iota} = \begin{bmatrix} \hat{e}_1 & & \mathbf{0} & \\ & \hat{e}_2 & \ddots & \\ \mathbf{0} & & & \hat{e}_{\alpha_l} \end{bmatrix},$$

where e is a column vector with a one in the first position and all other elements zero, and $\hat{e}$ is a row vector with a one in the last position and all other elements zero.

A system in Thorp-Morse form is composed of five decoupled subsystems, as follows:

(1) a subsystem corresponding to the infinite zero structure, and defined by $(A_{\infty\infty}, B_{\infty\infty}, C_{\infty\infty}, D_{\infty\infty})$. It is controllable and observable. It consists of decoupled chains of integrators with inputs and outputs.

(2) A subsystem corresponding to Kronecker column indices and defined by $(A_{\varepsilon\varepsilon}, B_{\varepsilon\varepsilon})$. This subsystem is controllable but unobservable. It consists of decoupled chains of integrators with inputs, but no outputs.

(3) A subsystem corresponding to Kronecker row indices and defined by $(C_{\eta\eta}, A_{\eta\eta})$. This subsystem is observable but now controllable. It consists of decoupled chains of integrators with outputs, but no inputs.

(4) A subsystem corresponding to the finite zero structure and defined totally by A_{ff}. This subsystem contains the finite zeros of the original system. It is completely uncontrollable and unobservable.

(5) A "feedforward" subsystem defined by $\mathrm{diag}\{(I)^{(r)}, 0^{(g)(h)}\}$. This subsystem passes r inputs unchanged to r outputs, annihilates h inputs, and generates g identically zero outputs.

The Thorp-Morse form is related to the Kronecker form by row and column exchanges and simple sign changes, which can be expressed as s.e. transformations. Thus the Thorp-Morse form is a canonical form for matrix pencils under strict equivalence. Thorp (1973) points out that the transformation of a system matrix structure to its Thorp-Morse form can be implemented via nonsingular coordinate transformations of the state, input, and output spaces, state feedback, and output injection.

Because the system matrix structure is preserved, the Thorp-Morse form is directly associated with a canonical control system. Techniques of control design and analysis can be applied directly to the canonical system, simplifying and clarifying the design process. Because the transformation to Thorp-Morse form has a feedback interpretation, it—or some approximation to it—may be incorporated into a compensator. Thus the Thorp-Morse form has potential applications in both analysis and design.

3.1. *On the Equivalence and Structural Stability of Zero Structures*

If the discussion so far, two systems have been considered to have equivalent zero structures if their system matrices were related by strict equivalence. This definition of equivalence in turn induces a definition for structural instability, namely that a control system has a structurally unstable zero structure if an arbitrarily small perturbation can change its Kronecker form. These are not entirely satisfactory definitions from the point of view of control-system design. Consider a square system with a well-defined relative degree vector of zeros. This system has only finite zeros, in number equal to that of the poles. Small perturbations, under the constraint that the perturbed system must be proper, will only slightly perturb the finite zeros. As long as these variations do not move a zero to or across the imaginary axis, they have little effect on compensator design. Leaving aside the issue of causality, which will be discussed in the next section, it seems that this system should be structurally stable. But, since two pencils are s.e. only if they have identical finite elementary divisors, it is not. For this reason, the following relaxed definition of equivalence is suggested.

Define two control systems to have equivalent zero structures if their system matrices have identical infinite zero structure and right and left singular structures, and if the numbers of finite zeros in the left half plane, on the imaginary axis but not at the origin, at the origin, and in the right half plane all coincide. Note that under this equivalence relation a square control system with relative degree zero and no zeros on the imaginary axis is structurally stable. This relaxed definition of equivalence, somewhat similar to the eigenvalue "bundles" considered by Arnold (1981), is discussed further in Section 5.

4. A CANONICAL UNFOLDING OF THE THORP-MORSE FORM

This section briefly discusses some aspects of the miniversal deformation presented in Berg and Kwatny (1995). It has two features that make it particularly suitable for control-system analysis and design. The first is that it has a *simple* structure, that is, each element of the base vector appears exactly once. The second is that it separates perturbations of the linear part of the pencil, called *noncausal perturbations*, from perturbations to the constant part of the pencil, called *causal perturbations*. The noncausal perturbations are entirely characterized by pure differentiators in the matrix D. Thus, the terminology is motivated by the physical interpretation of these terms. The separation of causal and noncausal parts allows the analyst to consider only parturbations that result in a realizable system. The miniversal deformation presented by Edelman, Elmroth, and Kagstrom (1995) lacks this feature. This drawback is particularly noticeable when considering questions of genericity. If causality is not enforced, the generic zero structures are composed entirely of singular Kronecker structures, giving an identically zero transfer matrix. With causality enforced, as will be seen, more practical stable structures may be obtained. The analysis of Ferrer and Garcia-Planas (1996) inherently enforces causality.

5. EXAMPLES

5.1. Regulating a Simple Parametrized Family of Linear Systems

This example is inspired by a problem studied by Kwatny, Bennett, and Berg (1990). The regulation of the flight-path angle of a relaxed-static-stability aircraft with uncertain parameters failed when the corresponding system pencil became singular under center-of-mass variation, corresponding to a saddle-node bifurcation. Examination of the system revealed that the two columns of B were linearly dependent at the bifurcation point. No single linear compensator can regulate the system for parameter values in a neighborhood of the bifurcation point (Berg and Kwatny, 1994). The following related linear problem exhibits many of the same points of interest:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.1}$$

$$\Gamma(s, \mu) = \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & \mu \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{5.2}$$

The control objective is regulation of the outputs despite step changes in the inputs (representing pilot commands) and slow variation, possibly large, in the parameter μ, which represents aircraft center-of-mass variation. A reasonable approach to this problem is through the theory of robust regulation (Francis, 1977; Kwatny and Kalnitsky, 1978). If a solution exists, it is guaranteed to handle the step commands as required. It is also guaranteed to handle small changes in the entries of B. It is not guaranteed to work if B undergoes a large perturbation, as may occur in this problem. Therefore some dependence on μ may need to be built into the compensator, which the miniversal deformation will be used to determine.

By Theorem 2 of Francis (1977), a structurally stable solution exists if and only if $\operatorname{rank}\Gamma(0, \mu) = n + p = 4$, that is, if and only if $\mu \neq 0$. So at the point $\mu = 0$, no solution exists. This point presents a problem to the designer. Clearly the control objectives cannot be satisfied when $\mu = 0$. But is it possible to design a single compensator to satisfy the control objectives everywhere else? Or perhaps this behavior is pathological, an artifact of the way the model has been constructed. Can the troublesome point be removed by slightly perturbing the model equations? To answer these questions, consider the canonical unfolding of the system matrix $\Gamma(0, \mu)$. The general unfolding is, for unconstrained systems,

$$\Gamma(s, \mathbf{c}) = \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & c_1 \\ -1 & 0 & c_6 s + c_2 & c_7 s + c_3 \\ 0 & -1 & c_8 s + c_4 & c_9 s + c_5 \end{bmatrix}. \tag{5.3}$$

Under these general conditions, the system $\Gamma(0, \mu)$ has codimension nine, and in fact most nearby one-parameter families will not contain this singularity. Therefore, if there are no other constraints on the system, the designer can consider a slightly perturbed model.

What happens if the system is constrained to be causal? The unfolding becomes

$$\Gamma(s, \mathbf{c}) = \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & c_1 \\ -1 & 0 & c_2 & c_3 \\ 0 & -1 & c_4 & c_5 \end{bmatrix}. \tag{5.4}$$

Now the only stable families containing the problematic structure depend on at least five parameters. So again in this case, the designer may conclude that setting the matrix $\boldsymbol{D}$ to $\boldsymbol{0}$ is an error in modeling, and that in practice only systems with an invertible $\boldsymbol{D}$ will occur. Assume now, and for the remainder of this problem, that it is physically realistic to enforce the condition that the perturbed system is strictly proper. The unfolding then is

$$\Gamma(s, \mathbf{c}) = \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & c \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{5.5}$$

In this case the original parametrization coincides with the unfolding. By the properties of the unfolding, then, the unstable structure associated with the case $\mu = c = 0$ will be persistent. Therefore, it is not an artifact of the modeling process, and must be considered in the design process. This is consistent with the results of Kwatny, Bennett, and Berg (1990), where it was observed in a computer simulation study involving only one parameter.

The unfolding can be used to draw a bifurcation diagram. The case $c = 0$ has Kronecker indices $\varepsilon_1 = 0$, $\eta_1 = 1$, $\rho_1 = 2$. The Thorp-Morse form corresponds to two decoupled integrators, one of which is uncontrollable. For $c \neq 0$, whether negative or positive.

$$\begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & c \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

with Kronecker indices $\rho_1 = 2$, $\rho_2 = 2$. The Thorp-Morse form corresponds to two decoupled integrators, both observable and controllable. Whether c is real or complex, there is a s.e. transformation to a system pencil with all real entries (the Thorp-Morse form, for example). By the same token, if c is restricted to be real, there is no loss of generality. So the parameter space is $\mathbb{R}^1$, and it can be partitioned into $\mathscr{C}^- = \{c : c < 0\}$, $\mathscr{C}^0 = \{c : c = 0\}$, $\mathscr{C}^+ = \{c : c > 0\}$. Since the structure on $\mathscr{C}^-$ and $\mathscr{C}^+$ is the same, and the closure of the union of these sets is the whole line, this structure is generic. Figure 5 shows the bifurcation diagram. The result differs from the usual bifurcation diagram for a dynamical system. In particular, the structures on either side of the bifurcation point are identical. However, with respect to the closed-loop system of definite bifurcation does occur.

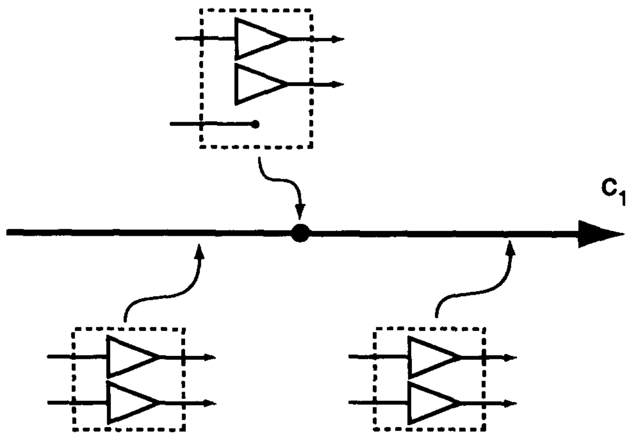

FIG. 5. Bifurcation diagram for system (5.1), (5.2).

A compensator can be designed at any point $c = \bar{\mu} \neq 0$ following Francis (1977). It is easy to find a compensator that satisfies the requirements for regulation for every c in either $\mathscr{C}^-$ or $\mathscr{C}^+$. It can be shown that no such compensator is also stabilizing on both $\mathscr{C}^-$ and $\mathscr{C}^+$. For any *fixed* compensator the closed-loop system will go unstable as c passes through the origin. Therefore the best that can be achieved is a compensator that regulates everywhere on either $\mathscr{C}^-$ or $\mathscr{C}^+$. So for the family of systems described by (5.1), (5.2), a two-member family of controllers is chosen by designing one at some $\bar{\mu}^+ > 0$ and the other at some $\bar{\mu}^- < 0$. One of these two compensators will regulate the system for any value of c except zero. This is the best possible result, in that no smaller family exists with this property.

5.2. *Structural Stability for Families Containing Singular Elements*

This section considers the unfolding of several other singular structures of related systems. In each case, all members of the parametrized family are required to be strictly proper. The following case is intuitively "less common" than the previous example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.6}$$

The canonically perturbed system, under the restrictions that the perturbed system must be causal and that there must be no feedforward term, is

$$\Gamma(s, \mathbf{c}) = \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s - c_1 & 0 & c_2 \\ -1 & 0 & 0 & 0 \\ 0 & -c_3 & 0 & 0 \end{bmatrix}. \tag{5.7}$$

Clearly c_2 and c_3 can be restricted to be real without loss of generality, and c_1 must be so restricted to ensure realizability, so set $\mathscr{C} = \mathbb{R}^3$. Because $\dim\{\mathscr{C}\} = 3$, parameterized families containing the singular case $\Gamma(s, \mathbf{0})$ require at least three parameters for this structure to be persistent under perturbation.

The singularity is now unfolded. For $c_1 = c_2 = c_3 = 0$ we recover the nominal system. For c_1 arbitrary, $c_2 \neq 0$, $c_3 = 0$,

$$\begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s - c_1 & 0 & c_2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For c_1 arbitrary, $c_2 = 0$, $c_2 \neq 0$,

$$\begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s - c_1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -c_3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

For c_1 arbitrary, $c_2 \neq 0$, $c_3 \neq 0$,

$$\begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s - c_1 & 0 & c_2 \\ -1 & 0 & 0 & 0 \\ 0 & -c_3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Finally, for c_1 arbitrary, $c_2 = 0$, $c_3 = 0$, we have

$$\begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s - c_1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The parameter space is $\mathbb{R}^3$, and the above considerations split it into quadrants. A feature of this case not seen in the example of Section 5.1 is the set of pencils $c_2 = 0$, $c_3 = 0$. This set forms a surface of codimension two and occurs at the intersection of two manifolds of codimension one. Those two manifolds are actually orbits under strict equivalence, and each is a single partition. The set on the intersection, however, is composed of a continuum of partitions, each containing a single point. Using the relaxed definition of equivalence suggested in Section 3, this set splits into three equivalence classes, $c_1 < 0$, $c_1 = 0$, $c_1 > 0$. The bifurcation diagram is shown in Figure 6.

The "most degenerate" system of this order is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.8}$$

This system has the canonically perturbed system matrix

$$\Gamma(s, c) = \begin{bmatrix} s - c_1 & 0 & c_2 & c_3 \\ 0 & s - c_4 & c_5 & c_6 \\ -c_7 & -c_8 & 0 & 0 \\ -c_9 & -c_{10} & 0 & 0 \end{bmatrix} \tag{5.9}$$

and so requires ten-parameter families for persistence. To ensure realizability, c_1 and c_4 must either be real or complex conjugate pair.

6. CONCLUSIONS

This paper applies an unfolding of the zero structure of finite-dimensional, linear, time-invariant control systems. The unfolding has been ar-

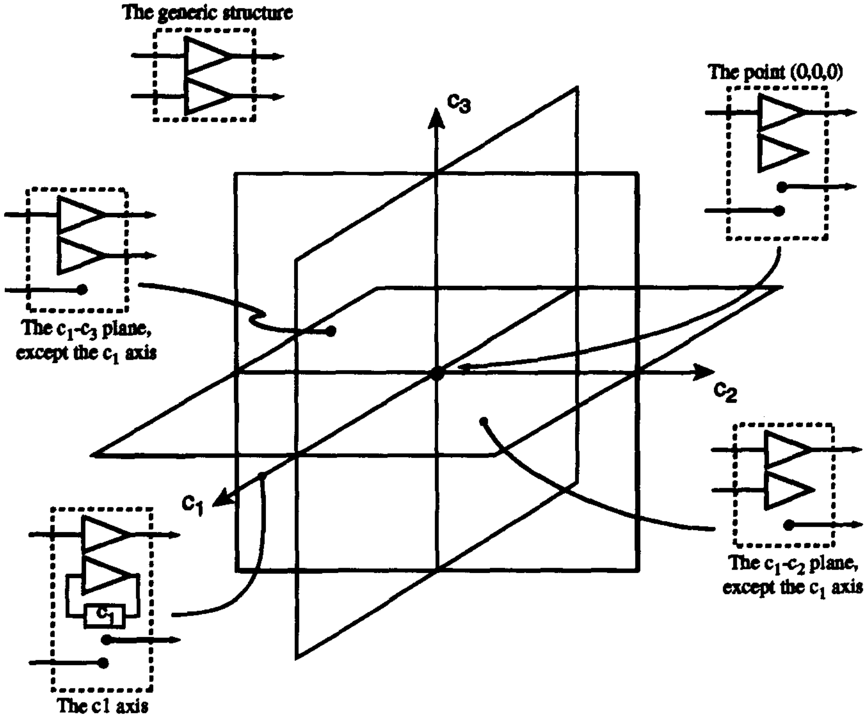

FIG. 6. Bifurcation diagram for system (5.7).

ranged so that common constraints, such as requiring the system to be proper or strictly proper, can be easily enforced.

The unfolding was applied in two ways. Several structurally unstable systems were examined, and their codimensions calculated. Based on calculations of this type, some structures can be shown to be artifacts of the modeling process. If the parametrized family containing the structure is of dimension less than the codimension of the structurally unstable member, then that member will never be observed in practice, and the designer can safely ignore it and work with a slightly perturbed model. Otherwise, the degenerate structure may be persistent, and the designer must allow for the singularity. One such example was presented, with the unfolding used to design a family of compensators. The family of compensators was, given the basic approach, the smallest possible capable of achieving the specified control objectives at all points where necessary conditions for those objectives were satisfied.

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